Saturation of 0-1 Matrices

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0-1 Matrices

Notation

For brevity: $\bullet$ denotes a 1 entry, and empty space denotes a 0.

Examples

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} \bullet & \bullet & & \bullet \\ & \bullet & & \\ \bullet & & \bullet & \\ & & \bullet & \bullet \end{pmatrix}$$

0-1 Matrices: Pattern Containment

Definition (Pattern Containment)

A 0-1 matrix M *contains* a 0-1 matrix P if M has a submatrix P' which can be changed to P by changing ones to zeroes.

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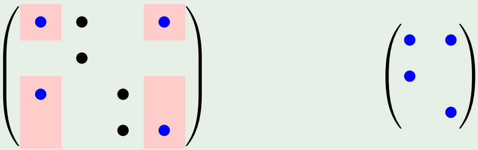

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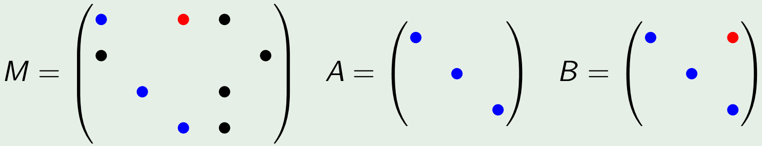

Saturation

Definition: Saturation

We say a matrix M is *saturated* for P, or P*-saturating*, if M does *not* contain P, but changing any 0 entry in M to 1 introduces a copy of P.

Example

If $P = I_2 = \begin{pmatrix} \bullet & \\ & \bullet \end{pmatrix}$, then

$$M = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & & & \\ \bullet & & & \end{pmatrix}$$

is saturated for P.

A Simple Problem

Example

Let $P = (\bullet\,\bullet)$. What is the minimum number of 1s in $m \times n$ P-saturating matrix?

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Solution

Every P-saturating matrix has exactly one 1 entry in each row, so the answer is m.

The Saturation Function

Definition: Saturation Function

The saturation function is defined to be the smallest number of 1s in an $m \times n$ P-saturating matrix, denoted by $\mathrm{sat}(m, n, P)$. If $m = n$, this is shortened to $\mathrm{sat}(n, P)$.

Example

$\mathrm{sat}\left(m, n, \begin{pmatrix} \bullet & \bullet \end{pmatrix}\right) = m.$

Example (Brualdi & Cao, 2020)

$\mathrm{sat}(m, n, I_k) = (k-1)(m+n-(k-1)).$

Saturation has also been applied to many other combinatorial structures, such as graphs, sequences, and posets.

Semisaturation

A useful lower bound for $\text{sat}(n, P)$ is given by:

Definition: Semisaturation

A matrix M is *semisaturated* for P if, whenever a 0 entry in M is changed to a 1, a new copy of P is introduced.
The *semisaturation function* $\text{ssat}(m, n, P)$ is the minimum number of ones in an $m \times n$ matrix that is semisaturated for P.

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Observation (Fulek and Keszegh, 2021)

In particular, since every saturated matrix for P is also semisaturated for P, we have the inequality

$$\mathrm{ssat}(m, n, P) \leq \mathrm{sat}(m, n, P).$$

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Example (Semisaturation)

The matrix

$$\begin{pmatrix} \bullet & & & \bullet \\ & & & \\ & & & \\ \bullet & & & \bullet \end{pmatrix}$$

is semisaturating for $P = I_2 = \begin{pmatrix} \bullet & \\ & \bullet \end{pmatrix}$

Dichotomy

In general, determining the exact value of $\text{sat}(m, n, P)$ is very difficult. Usually, we only care about the growth rate.

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Theorem (Fulek and Keszegh, 2021)

For any pattern P, either $\text{sat}(n, P) = \Theta(n)$ or $\text{sat}(n, P) = O(1)$. Additionally, for m_0, n_0 fixed, $\text{sat}(m_0, n, P)$ is either $\Theta(n)$ or $O(1)$, and $\text{sat}(m, n_0, P)$ is either $\Theta(m)$ or $O(1)$.

This poses the question: which matrices have a linear saturation function, and which have a bounded saturation function?

Criteria for Patterns with Linear Saturation Functions

Several sufficient conditions have been found for patterns P with the property that $\mathrm{sat}(n, P) = \Theta(n)$.

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Theorem (Fulek and Keszegh, 2021)

P has linear saturation function if the first or last row or column of P contains no 1-entries.

Criteria for Patterns with Linear Saturation Functions

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Theorem (Fulek and Keszegh, 2021)

P has linear saturation function if it is of the form $\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}$ or $\begin{pmatrix} \mathbf{0} & A \\ B & \mathbf{0} \end{pmatrix}$, for nonzero matrices A and B. Such matrices are called decomposable.

Criteria for Patterns with Bounded Saturation Functions

Theorem (Geneson, 2021)

Almost all permutation matrices have bounded saturation functions.

Theorem (Berendsohn, 2023)

A permutation matrix has a bounded saturation function if and only if it is indecomposable.

Expandable Rows and Columns

Definition: Expandable Row

An *expandable row* for a matrix M with respect to a pattern P is defined as an all-zero row for which exchanging any zero in the row for a one will create a new occurrence of P in M. The definition is similar for *expandable columns*.

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Example

If $P = \begin{pmatrix} \bullet & \bullet & \bullet \end{pmatrix}$, then in matrix

$$M = \begin{pmatrix} \bullet & \cdot & \bullet \\ \bullet & \cdot & \bullet \\ \bullet & \cdot & \bullet \end{pmatrix},$$

the second column is expandable with respect to P.

Witnesses

Definition: Witness

A matrix M is a *witness* for pattern P if and only if M does not contain P and M contains an expandable row and column.

Analogous definitions are given for *horizontal* and *vertical* witnesses, corresponding to P-free matrices containing an expandable column, and P-free matrices containing an expandable row, respectively.

Witnesses

Witnesses are useful to consider because of the following result:

Theorem (Fulek and Keszegh, 2021; Berendsohn, 2021)

There exists a witness for a pattern P if and only if $\mathrm{sat}(n, P) = O(1)$.

Witness Example

We proceed with an example of this property being used to prove that a specific pattern P has $\mathrm{sat}(n, P) = O(1)$.

Example (Berendsohn, 2021)

Claim: The matrix

$$P = \begin{pmatrix} & & \bullet & \\ \bullet & & & \\ & & & \bullet \\ & \bullet & & \end{pmatrix}$$

has the property that $\mathrm{sat}(n, P) = O(1)$.

Witness Example

Example (Berendsohn, 2021)

Consider the matrix

$$W = \begin{pmatrix}
 & & & & & & & & \cdot & \bullet & \\
 & & & & & & & \bullet & \cdot & & \\
 & & & & & & & & \cdot & & \bullet \\
 & & & & & & \bullet & & \cdot & & \\
 & & & & & & & & \cdot & \bullet & \\
 & & & & & & & \bullet & \cdot & & \\
 & & \bullet & & & & & & \cdot & & \\
\bullet & & & & \bullet & & & & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \bullet & & & & \bullet & & & \cdot & & \\
 & & & \bullet & & & & & \cdot & &
\end{pmatrix}.$$

Because W is a witness for P, $\mathrm{sat}(n, P) = O(1)$.

Fixing One Dimension

It is easy to show that $\text{sat}(m_0, n, P) = O(1)$ and $\text{sat}(m, n_0, P) = O(1)$ are both necessary conditions for $\text{sat}(n, P) = O(1)$. When are they sufficient?

Fixing One Dimension

Theorem (Berendsohn, 2021)

Let P be an indecomposable pattern with no empty rows or columns, with only one 1 entry in the last row and only one 1 entry in the last column. The following are equivalent:

- *There exists both a horizontal witness W_H and a vertical witness W_V for P.*
- $\text{sat}(m_0, n, P) = O(1)$ *and* $\text{sat}(m, n_0, P) = O(1)$
- $\text{sat}(n, P) = O(1)$.

Fixing One Dimension—Generalizations

The question on the relationship for further classes of matrices remains difficult to fully resolve. We have proven, however, that the simple existence of horizontal and vertical witnesses does not imply boundedness in the general case.

Fixing One Dimension—Generalizations

Shown below is an example of a matrix that has both horizontal and vertical witnesses, but has linear saturation function.

Example

Let

$$P = \begin{pmatrix} \bullet & \bullet & & \\ & & & \bullet \\ \bullet & & & \\ & & \bullet & \bullet \end{pmatrix}.$$

Then $\mathrm{sat}(m_0, n, P) = O(1)$ and $\mathrm{sat}(m, n_0, P) = O(1)$, but $\mathrm{sat}(n, P) = \Theta(n)$.

Fixing One Dimension—Strong Indecomposability

We extend Berendsohn's theorem to provide a necessary/sufficient condition for certain patterns.

Theorem

If an indecomposable pattern P is not of the form

$$P = \begin{pmatrix} \mathbf{0} & P_1 & \mathbf{0} \\ P_2 & \mathbf{0} & P_3 \\ \mathbf{0} & P_4 & \mathbf{0} \end{pmatrix},$$

then $\operatorname{sat}(n, P) = O(1)$ *if and only if there exist vertical and horizontal witnesses with the property that at least one of the zeroes in each of the respective empty rows and columns correspond to a* 1 *in* P *that is alone in its row and column. Such matrices are defined as strongly indecomposable.*

Fixing One Dimension—Strong Indecomposability

Theorem

If an indecomposable pattern P is strongly indecomposable, then $\mathrm{sat}(n, P) = O(1)$ if and only if there exist vertical and horizontal witnesses with the property that at least one of the zeroes in each of the respective empty rows and columns correspond to a 1 in P that is alone in its row and column.

It is likely that the strong indecomposability condition can be loosened in certain cases, and the question for non-strongly indecomposable matrices remains open.

Fixing One Dimension—Other Results

We have established several more results for the one-dimensional case.

Theorem

For a pattern P and sufficiently large m_0, the semisaturation function $\mathrm{ssat}(m_0, n, P) = O(1)$ if and only if the top and bottom rows of P each contain a 1 entry which is the only 1 entry in its column.

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Theorem

If a $k \times l$ pattern P has no empty columns and $\mathrm{sat}(m_0, n, P) = O(1)$, then $\mathrm{sat}(m_0, n, P) \leq (l-1)m_0^2$, and there is a horizontal witness for P with at most $(l-1)m_0 + 1$ columns.

Adding Empty Rows and Columns

We have some results involving empty rows and saturation:

Theorem

Let P have $\text{sat}(m_0, n, P) = O(1)$, and P' be derived from P by adding empty columns. Then $\text{sat}(m_0, n, P') = O(1)$.

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Theorem

Let P be a permutation matrix with $\mathrm{sat}(n, P) = O(1)$, and let P' be derived from P by adding empty rows and columns in the interior (i.e., not before the first or after the last). Then $\mathrm{sat}(n, P') = O(1)$.

More Patterns

Finally, we have also found the saturation functions for several specific patterns, some of which can be generalized to infinite families.

Theorem

Let

$$Q_2 = \begin{pmatrix} \bullet & & \bullet & \\ \bullet & & & \\ & & & \bullet \\ & \bullet & & \end{pmatrix}, \qquad Q_3 = \begin{pmatrix} & \bullet & & \\ \bullet & & & \\ & & & \bullet \\ \bullet & & \bullet & \end{pmatrix},$$

$$Q_4 = \begin{pmatrix} \bullet & & \bullet & \\ \bullet & & & \\ & & & \bullet \\ \bullet & \bullet & & \end{pmatrix}, \qquad Q_5 = \begin{pmatrix} & \bullet & & \\ \bullet & & & \\ & & & \bullet \\ \bullet & & \bullet & \bullet \end{pmatrix}.$$

Then Q_2 and Q_4 have bounded saturation function, while Q_3 and Q_5 have linear saturation function.

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